

# Landau's necessary density conditions for the Hankel transform

Luís Daniel Abreu\*

Afonso S. Bandeira†

November 30, 2011

## Abstract

We will prove an analogue of Landau's necessary conditions [*Necessary density conditions for sampling and interpolation of certain entire functions*, Acta Math. 117 (1967).] for spaces of functions whose Hankel transform is supported in a measurable subset  $S$  of the positive semi-axis. As a special case, necessary density conditions for the existence of Fourier-Bessel frames are obtained.

**Keywords:** Sampling and Interpolation, Beurling-Landau density, Hankel transform, Bessel functions, Fourier-Bessel frames.

## 1 Introduction

While Fourier Series rely on the fact that  $\{e^{ikx}\}_{k \in \mathbb{Z}}$  constitutes an orthogonal basis for  $L^2(-\pi, \pi)$ , *Nonharmonic Fourier Series* allow more general sets  $\{e^{it_k x}\}_{k \in \mathbb{Z}}$ . They can be nonuniform as in *Riesz Basis* [21], perhaps even redundant as in *Fourier Frames* [13]. On their “frequency side”, nonharmonic Fourier series provide nonuniform and redundant sampling theorems in spaces of bandlimited functions. As a consequence of Landau's necessary conditions for sampling and interpolation of such functions [10, 11], we know that sampling requires  $\{t_k\}_{k \in \mathbb{Z}}$  to be “denser than  $\mathbb{Z}$ ” and that interpolation requires  $\{t_k\}_{k \in \mathbb{Z}}$  to be “sparser than  $\mathbb{Z}$ ”. The set  $\mathbb{Z}$  is a sequence of both sampling and interpolation for bandlimited functions (this is known as the Whittaker-Shannon-Kotel'nikov sampling theorem).

Likewise, let  $J_\alpha$  be the Bessel function of order  $\alpha > -1/2$  and  $j_{n,\alpha}$  its  $n^{\text{th}}$  zero. Several classical results in Fourier analysis have been extended to Fourier-Bessel series [3, 4, 8, 20]. The theory of Fourier-Bessel series is based on the fact that  $\{x^{\frac{1}{2}} J_\alpha(j_{n,\alpha} x)\}_{n=0}^\infty$  is an orthogonal basis for  $L^2[0, 1]$ . Thus, the study of more general sets  $\{x^{\frac{1}{2}} J_\alpha(t_n x)\}_{n=0}^\infty$  leads naturally to “nonharmonic Fourier-Bessel sets”. Completeness properties of such sets have been investigated by Boas and Pollard [2], and some stability results concerning Riesz basis have been obtained

---

\*Department of Mathematics of University of Coimbra, 3001-454 Coimbra ([daniel@mat.uc.pt](mailto:daniel@mat.uc.pt)). Currently at NuHAG, University of Vienna, in (FWF) project “Frames and Harmonic Analysis”. This research was partially supported by CMUC/FCT and FCT project “Frame Design” PTDC/MAT/114394/2009, POCI 2010 and FSE.

†Program in Applied and Computational Mathematics, Princeton University, NJ 08544, USA ([ajsb@math.princeton.edu](mailto:ajsb@math.princeton.edu)). Part of this work was done while the second author was at Department of Mathematics of University of Coimbra supported by the research grant BII/FCTUC/C2008/CMUC. Partially supported by CMUC/FCT and FCT project “Frame Design” PTDC/MAT/114394/2009, POCI 2010 and FSE.

in [17]. However, the problems that arise naturally in connection with frame theory and, in particular, questions related to frame and sampling density, have not been investigated up to the present date. We will address this question in the present paper, as a special case of a more general result, which gives Landau-type results in the context of the Hankel transform.

Throughout this paper  $S$  is assumed to be a measurable subset of  $(0, \infty)$ .

Consider the space  $\mathcal{B}_\alpha(S)$  of functions in  $L^2(0, \infty)$  such that their Hankel transform,

$$H_\alpha(f)(x) = \int_0^\infty f(t)(xt)^{1/2} J_\alpha(xt) dt,$$

is supported in  $S$ . The special case  $S = [0, 1]$  is an important example of a reproducing kernel Hilbert space with an associated sampling theorem [9]. Moreover, this reproducing kernel Hilbert space is strongly reminiscent of the classical Paley-Wiener space of bandlimited functions. In particular, with a view to solving an eigenvalue problem arising in the theory of random matrices, Tracy and Widom [19] have constructed a set of functions which play the role of the prolate spheroidal functions in this situation. Such functions are examples of doubly orthogonal functions in the sense of Stefan Bergman [1]. This automatically implies [18] that they solve the concentration problem

$$\lambda_k \phi_k(x) = \int_0^r \phi_k(t) \mathcal{R}_\alpha(t, x) dt,$$

where  $\mathcal{R}_\alpha(t, x)$  is the reproducing kernel of  $\mathcal{B}_\alpha([0, 1])$ . Once we know that such functions exist, it becomes natural to ask if the behaviour of the corresponding eigenvalues displays the “plunging phenomenon” which has been observed in association with the “Nyquist rates” described in terms of Beurling-type density theorems (see [10], [18] and the discussion in [5, Chapter 2]). We will see that this is indeed the case, even in the more general case of the space  $\mathcal{B}_\alpha(S)$ , where  $S$  is a measurable subset of the positive semi-axis.

The description of our results requires some terminology. A sequence  $\Lambda = \{t_n\}_{n=0}^\infty$  is a *set of sampling* for  $\mathcal{B}_\alpha(S)$  if there exists a constant  $A$  such that, for every  $f \in \mathcal{B}_\alpha(S)$ ,

$$A \int_0^\infty |f(x)|^2 dx \leq \sum_{n=0}^\infty |f(t_n)|^2.$$

Moreover,  $\Lambda$  is a *set of interpolation* for  $\mathcal{B}_\alpha(S)$  if, given any set of numbers  $\{a_n\}_{n=0}^\infty$  with  $\sum |a_n|^2 < \infty$ , there exists  $f \in \mathcal{B}_\alpha(S)$  such that

$$f(t_n) = a_n, \text{ for every } t_n \in \Lambda.$$

We say that a sequence is separated if the distance between any two distinct points exceeds some positive quantity  $d > 0$ . For such sequences we can define densities which are suitable for analysis of functions supported in  $(0, \infty)$ .

**Definition 1** Let  $n_a(r)$  denote the number of points of  $\Lambda \subset (0, \infty)$  to be found in  $[a, a + r]$ . Then the lower and the upper densities of  $\Lambda$  are given by the limits

$$D^-(\Lambda) = \lim_{r \rightarrow \infty} \inf_{a \geq 0} \frac{n_a(r)}{r} \quad \text{and} \quad D^+(\Lambda) = \lim_{r \rightarrow \infty} \sup_{a \geq 0} \frac{n_a(r)}{r}.$$

Our main results read as follows.

**Theorem 1** *Let  $S$  be a measurable subset of  $(0, \infty)$  and  $\alpha > -1/2$ . If a separated set  $\Lambda$  is of sampling for  $\mathcal{B}_\alpha(S)$ , then*

$$D^-(\Lambda) \geq \frac{1}{\pi} m(S). \quad (1)$$

**Theorem 2** *Let  $S$  be a bounded measurable subset of  $(0, \infty)$  and  $\alpha > -1/2$ . If the set  $\Lambda$  is of interpolation for  $\mathcal{B}_\alpha(S)$ , then*

$$D^+(\Lambda) \leq \frac{1}{\pi} m(S). \quad (2)$$

A major technical difficulty in the proofs of the above results arises from the translation invariance of Definition 1, since we cannot appeal to the translation invariance of the eigenvalue problem which was used by Landau in [10]. For this reason, delicate estimates of operators involving the reproducing kernels of the space  $B_\alpha([a, a+r])$  are required.

We will also prove that the separation condition implies the existence of a constant  $B$  such that, for every  $f \in B_\alpha(S)$ ,

$$\sum_n |g(t_n)|^2 \leq B \|g\|^2.$$

Theorem 1 can be seen from the *frame theory* viewpoint. A sequence of functions  $\{e_j\}_{j \in I}$  is said to be a *frame* in a Hilbert space  $H$  if there exist positive constants  $A$  and  $B$  such that, for every  $f \in H$ ,

$$A \|f\|_H^2 \leq \sum_{j \in I} |\langle f, e_j \rangle|^2 \leq B \|f\|_H^2. \quad (3)$$

Accordingly, we say that  $\{(t_n x)^{\frac{1}{2}} J_\alpha(t_n x)\}$  is a *Fourier-Bessel frame* if there exist positive constants  $A$  and  $B$  such that, for every  $f \in \mathcal{B}_\alpha[(0, 1)]$ ,

$$A \int_0^1 |f(x)|^2 dx \leq \sum_{n=0}^{\infty} \left| \int_0^1 (t_n x)^{\frac{1}{2}} f(x) J_\alpha(t_n x) dx \right|^2 \leq B \int_0^1 |f(x)|^2 dx.$$

By choosing  $S = (0, 1)$  in Theorem 1 one concludes that, if  $\{(t_n x)^{\frac{1}{2}} J_\alpha(t_n x)\}$  is a Fourier-Bessel frame, then  $D^-(\Lambda) \geq \frac{1}{\pi}$ . In particular, the orthogonal basis  $\{(j_{n,\alpha} x)^{\frac{1}{2}} J_\alpha(j_{n,\alpha} x)\}$  is a Fourier-Bessel frame, since the norm of each element is bounded away from zero and infinity [8, (2.3)]. It is well known (as a consequence of the Paley-Wiener theorem: see [9, Theorem 2]) that if  $f \in \mathcal{B}_\alpha[(0, 1)]$  then  $t^{-\alpha-\frac{1}{2}} f(t)$  belongs to the Paley-Wiener space. Thus, every sufficient condition for Fourier frames also holds in the case of Fourier-Bessel frames. For an account of such conditions see, for instance, those in [13, pg. 791] and the references therein. Such an observation may be useful in the construction of the “Bessel analogues” of the hyperbolic lattice in [16, Theorem 3.4]

Recently, Marzo [12] applied Landau’s ideas to the proof of Marcinkiewicz–Zygmund inequalities in the sphere. From his work we borrow an idea to start with the estimations leading to inequality (1) and a method to deal with the case where a sequence of both sampling and interpolation is unknown (as in our more general situation) or do not exist (the case for higher dimensions in [12]). It is worth noting that analogues of Landau’s necessary conditions have been also studied [7, 6] using techniques from time-frequency analysis [14].

The outline of the paper is as follows. In Section 2 we collect some results about the convolution structure associated with the Hankel transform. The key section is Section 3, where

the eigenvalue problem is formulated and the estimates of the trace and norm are obtained. Section 4 contains the lemmas which are required, in the proofs of the main results, to establish the connection between the sampling and interpolation concepts and the eigenvalue problem. We prove our main results in Section 5.

## 2 Bessel functions and their convolution structure

In this section we will use [15] and [16] as reference sources for some definitions and properties that are useful in the harmonic analysis associated with the Hankel transform. For  $\alpha > -1$ , the *Bessel functions* are defined by the power series,

$$J_\alpha(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{x}{2}\right)^{2n+\alpha}}{n! \Gamma(n + \alpha + 1)}.$$

Bessel functions are solutions of the second order differential equation

$$\frac{d^2 y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \left(1 - \frac{\alpha^2}{x^2}\right) y = 0. \quad (4)$$

The derivative of a Bessel function can be related to a Bessel function of different order via the formula

$$\frac{1}{x^m} \left(\frac{d}{dx}\right)^m J_\alpha(x) = x^{\alpha-m} J_{\alpha-m}(x), \quad (5)$$

valid for every positive integer  $m$ . We will make extensive use of the asymptotic formulae [20]:

$$J_\alpha(x) = \sqrt{\frac{2}{\pi x}} (\sin \eta_x + \rho(x)), \quad (6)$$

with  $\rho(x) = \mathcal{O}(x^{-1})$ , where  $\eta_x = x - \left(\frac{1}{2}\alpha - \frac{1}{4}\right)\pi$ , and

$$J'_\alpha(x) = \sqrt{\frac{2}{\pi x}} (\cos \eta_x + \rho_1(x)) \quad (7)$$

with  $\rho_1(x) = \mathcal{O}(x^{-1})$ . Sometimes it is convenient to renormalize the Bessel functions in the following way:

$$j_\alpha(x) = \Gamma(\alpha + 1) \left(\frac{2}{x}\right)^\alpha J_\alpha(x).$$

The functions  $j_\alpha$  are the *spherical* Bessel functions. They satisfy  $j_\alpha(0) = 1$  and  $|j_\alpha(x)| \leq 1$ , for all  $x \in (0, \infty)$ . For the Harmonic Analysis associated with the Hankel transform one defines a “Hankel modulation”  $(m_\lambda f)(x) = j_\alpha(\lambda x)f(x)$  and associates with it a “Hankel translation”  $H_\alpha(\tau_\lambda f)(x) = (m_\lambda H_\alpha f)(x) = j_\alpha(\lambda x)(H_\alpha f)(x)$ . This allows to define a “*Hankel convolution*” as follows:

$$f *_\alpha g(\lambda) = \lambda^{\alpha+\frac{1}{2}} \left(\frac{\pi}{2}\right)^{\frac{\alpha}{2}} \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(t) \tau_\lambda g(t) dt.$$

Hankel convolutions are mapped in products via the formula

$$H_\alpha(f *_\alpha g)(x) = x^{-(\alpha+\frac{1}{2})} (2\pi)^{\frac{\alpha}{2}} H_\alpha f(x) H_\alpha g(x).$$

The following property of Hankel translations, which can be found, for instance, in [15], will be also required: if  $\text{supp } g \subset [0, d]$  and  $r > d$  then

$$\text{supp } \tau_r g \subset [\max\{0, r - d\}, r + d]. \quad (8)$$

### 3 The eigenvalue problem

Let  $S$  be a finite union of intervals and  $I$  be the interval  $I = [a, a + r]$ .

Let  $D(I)$  be the subspace of  $L^2(0, \infty)$  consisting of functions supported on  $I$  and  $\chi_I$  the characteristic function of  $I$ . Let  $D_I$  and  $B_S$  denote the orthogonal projections of  $L^2(0, \infty)$  onto  $D(I)$  and  $\mathcal{B}_\alpha(S)$ , respectively. They are given explicitly by

$$D_I f = \chi_I f \quad \text{and} \quad B_S f = H_\alpha D_S H_\alpha f$$

We want to maximize, over the functions  $f \in \mathcal{B}_\alpha(S)$ , the “energy concentration”  $\lambda_f$  given as

$$\lambda_f = \frac{\int_I |f(t)|^2 dt}{\|f\|^2}.$$

This is a standard problem of maximizing a quadratic form and leads to the eigenvalue problem

$$\lambda_k(I, S) \phi_k(x) = B_S D_I \phi_k. \quad (9)$$

Writing the operators explicitly and interchanging the integrals, (9) becomes

$$\lambda_k(I, S) \phi_k(x) = \int_I \phi_k(t) w_S(t, x) dt, \quad (10)$$

where, for a set  $X$ , the Reproducing Kernel  $w_X(t, x)$  is given by

$$w_X(t, x) = \int_X J_\alpha(ts) J_\alpha(xs) (tx)^{\frac{1}{2}} s ds. \quad (11)$$

Multiplying both sides of (10) by  $(xu)^{\frac{1}{2}} J_\alpha(xu)$ , integrating with respect to  $dx$  in  $I$  and changing the order of the integrals, gives the dual problem of concentrating on  $S$  functions whose Hankel Transform is supported on  $I$ :

$$\lambda_k(I, S) \psi_k(t) = \int_S \psi_k(x) w_I(x, t) dx. \quad (12)$$

Using this duality and a change of variables gives, for  $\beta > 0$ , the identities

$$\lambda_k(I, S) = \lambda_k(S, I) \quad (13)$$

$$= \lambda_k(\beta I, \beta^{-1} S). \quad (14)$$

Now set

$$\mathcal{R}_\alpha(t, x) = w_{[0,1]}(t, x) = \begin{cases} (tx)^{\frac{1}{2}} \frac{J_\alpha(t) x J'_\alpha(x) - J_\alpha(x) t J'_\alpha(t)}{t^2 - x^2} & \text{if } t \neq x \\ \frac{1}{2} (x J'_\alpha(x)^2 - x J_\alpha(x) J''_\alpha(x) - J_\alpha(x) J'_\alpha(x)) & \text{if } t = x. \end{cases} \quad (15)$$

and observe that

$$w_{[a,a+r]}(t, x) = (a+r)\mathcal{R}_\alpha((a+r)t, (a+r)x) - a\mathcal{R}_\alpha(at, ax). \quad (16)$$

We will first study the case when  $S$  is a finite union of intervals. Suppose  $S$  to consist of  $n$  disjoint intervals  $(b_1, b_1 + s_1), \dots, (b_n, b_n + s_n)$  and write  $s = s_1 + \dots + s_n$ . As in [10], the cornerstone of the proofs consists of Norm and Trace estimates of the above operators. From the above considerations one has

$$\begin{aligned} \text{Trace} &= \sum \lambda_k(I, S) = \int_S w_{[a,a+r]}(x, x) dx \\ &= \int_S (a+r)\mathcal{R}_\alpha((a+r)x, (a+r)x) - a\mathcal{R}_\alpha(ax, ax) dx \end{aligned} \quad (17)$$

$$= \sum_{i=1}^n \int_{b_i}^{b_i+s_i} (a+r)\mathcal{R}_\alpha((a+r)x, (a+r)x) - a\mathcal{R}_\alpha(ax, ax) dx, \quad (18)$$

and

$$\begin{aligned} \text{Norm} &= \sum \lambda_k^2(I, S) = \int_S \int_S w_{[a,a+r]}^2(t, x) dt dx \\ &= \int_S \int_S [(a+r)\mathcal{R}_\alpha((a+r)t, (a+r)x) - a\mathcal{R}_\alpha(at, ax)]^2 dt dx \end{aligned} \quad (19)$$

$$= \sum_{i=1}^n \sum_{j=1}^n \int_{b_i}^{b_i+s_i} \int_{b_j}^{b_j+s_j} [(a+r)\mathcal{R}_\alpha((a+r)t, (a+r)x) - a\mathcal{R}_\alpha(at, ax)]^2 dt dx. \quad (20)$$

### 3.1 Estimation of Trace

In this Section we will obtain the following estimation.

**Lemma 1** *For  $\alpha > -1/2$ ,  $\text{Trace} = \frac{1}{\pi}rs + \mathcal{O}(1)$ . More precisely, there exists a constant  $L$  such that for all positive  $a, r, b, s$  we have*

$$\left| \text{Trace} - \frac{1}{\pi}rs \right| \leq L. \quad (21)$$

**Proof.** We will first estimate the function

$$T(u) = \int_0^u \mathcal{R}_\alpha(x, x) dx = \frac{1}{2} \int_0^u (xJ'_\alpha(x)^2 - xJ_\alpha(x)J''_\alpha(x) - J_\alpha(x)J'_\alpha(x)) dx \quad (22)$$

For small  $x$ , the power series expansion of the Bessel function gives

$$J_\alpha(x) = \frac{x^\alpha}{2^\alpha \Gamma(\alpha + 1)} + \mathcal{O}(x^{\alpha+2}),$$

leading, for small  $u$ , to the estimate

$$T(u) = \frac{1}{2} \int_0^u \mathcal{O}(x^{2\alpha+1}) dx.$$

Therefore, for  $\alpha > -1/2$  and small  $u$ , the integral defining  $T(u)$  is convergent. We proceed to estimate  $T(u)$ . Using (6), (7) and (15) one obtains, after some simplification,

$$\mathcal{R}_\alpha(x, x) = \frac{1}{\pi} + \epsilon(x), \quad (23)$$

with  $|\epsilon(x)| = \mathcal{O}(\frac{1}{x})$ . It is possible (and it will be required for our purposes) to improve this estimate even more, taking into account the cancelations resulting from the changes in sign of  $\epsilon(x)$ . Use the second order differential equation (4) to rewrite  $\mathcal{R}_\alpha(x, x)$  as

$$\begin{aligned} \mathcal{R}_\alpha(x, x) &= \frac{1}{2} \left( xJ'_\alpha(x)^2 - xJ_\alpha(x) \left( -\frac{1}{x}J'_\alpha(x) - J_\alpha(x) + \frac{\alpha^2}{x^2}J_\alpha(x) \right) - J_\alpha(x)J'_\alpha(x) \right) \\ &= \frac{1}{2} \left( xJ_\alpha(x)^2 + xJ'_\alpha(x)^2 - \frac{\alpha^2}{x}J_\alpha(x)^2 \right) \\ &= \frac{1}{2} (xJ_\alpha(x)^2 + xJ'_\alpha(x)^2) + \mathcal{O}\left(\frac{1}{x^2}\right) \\ &= \frac{1}{2} \left( xJ_\alpha(x)^2 + x \left( J_{\alpha-1}(x) - \frac{\alpha}{x}J_\alpha(x) \right)^2 \right) + \mathcal{O}\left(\frac{1}{x^2}\right) \\ &= \frac{1}{2}xJ_\alpha(x)^2 + \frac{1}{2}xJ_{\alpha-1}(x)^2 - \alpha J_{\alpha-1}(x)^2 J_\alpha(x) + \mathcal{O}\left(\frac{1}{x^2}\right). \end{aligned}$$

The third and fifth equalities in the above calculation were obtained using (6) and the fourth one using (5). Now observe that  $\int_0^u xJ_\alpha(x)^2 dx = \mathcal{R}_\alpha(u, u) = \frac{1}{\pi} + \mathcal{O}(\frac{1}{u})$ , and that the same is true if one replaces  $\alpha$  by  $\alpha - 1$ . Moreover, from (6) we obtain  $\int_0^u J_{\alpha-1}(x)^2 J_\alpha(x) dx = \mathcal{O}(1)$ . We conclude that

$$T(u) = \frac{1}{\pi}u + \mathcal{O}(1). \quad (24)$$

Finally,

$$\begin{aligned} \text{Trace} &= \sum_{i=1}^n \int_{b_i}^{b_i+s_i} (a+r)\mathcal{R}_\alpha((a+r)x, (a+r)x) - a\mathcal{R}_\alpha(ax, ax) dx \\ &= \sum_{i=1}^n \left( \int_{b_i}^{b_i+s_i} (a+r)\mathcal{R}_\alpha((a+r)x, (a+r)x) dx - \int_{b_i}^{b_i+s_i} a\mathcal{R}_\alpha(ax, ax) dx \right) \\ &= \sum_{i=1}^n \left( \int_{(a+r)b_i}^{(a+r)(b_i+s_i)} \mathcal{R}_\alpha(x, x) dx - \int_{ab_i}^{a(b_i+s_i)} \mathcal{R}_\alpha(x, x) dx \right) \\ &= \sum_{i=1}^n ((a+r)s_i - as_i + \mathcal{O}(1)) = \frac{1}{\pi}rs + \mathcal{O}(1), \end{aligned}$$

entering the estimate (24) in the fourth identity. This is the required result. ■

### 3.2 Estimation of Norm

This section contains the key step, which is the estimation of

$$\text{Norm} = \sum_{i,j}^n \int_{b_i}^{b_i+s_i} \int_{b_j}^{b_j+s_j} [(a+r)\mathcal{R}_\alpha((a+r)t, (a+r)x) - a\mathcal{R}_\alpha(at, ax)]^2 dt dx. \quad (25)$$

**Proposition 1** *If  $\alpha > -1/2$ , the function Norm satisfies the estimate*

$$\text{Norm} \geq \frac{1}{\pi}rs - K \log(r) - L. \quad (26)$$

for some constants  $K$  and  $L$  not depending on  $r$  or  $a$ .

### 3.2.1 Preparation Lemmas

We divide the technical parts of the proof of Theorem 1 in a few preparation Lemmas.

**Lemma 2** *There exists a constant  $L$  such that, for any positive  $b, s, a, r$ , we have the following*

$$\int_a^{a+r} \int_0^\infty [(b+s)\mathcal{R}_\alpha((b+s)t, (b+s)x) - b\mathcal{R}_\alpha(bt, bx)]^2 dt dx \geq \frac{1}{\pi}rs - L.$$

**Proof.** The result will be derived by proving that

$$\int_a^{a+r} \int_0^\infty [(b+s)\mathcal{R}_\alpha((b+s)t, (b+s)x) - b\mathcal{R}_\alpha(bt, bx)]^2 dt dx \quad (27)$$

is equal to

$$\int_a^{a+r} (b+s)\mathcal{R}_\alpha((b+s)x, (b+s)x) - a\mathcal{R}_\alpha(bx, bx) dx. \quad (28)$$

and then using Lemma 1 to bound (28). In order to prove that (27) is equal to (28) we first notice that, if  $a \leq b$  then

$$\int_0^\infty ab\mathcal{R}_\alpha(ax, at)\mathcal{R}_\alpha(bx, bt)dt = b \int_0^\infty \frac{a}{b}\mathcal{R}_\alpha\left(\frac{a}{b}z, \frac{a}{b}t\right)\mathcal{R}_\alpha(z, t)dt,$$

where  $z = bx$ . Since  $\frac{a}{b}\mathcal{R}_\alpha\left(\frac{a}{b}z, \frac{a}{b}t\right)$  is the reproducing kernel of  $\mathcal{B}_\alpha(0, \frac{a}{b})$ ,  $t \rightarrow \frac{a}{b}\mathcal{R}_\alpha\left(\frac{a}{b}z, \frac{a}{b}t\right)$  is a function in  $\mathcal{B}_\alpha(0, \frac{a}{b})$  and thus in  $\mathcal{B}_\alpha(0, 1)$ , since  $\frac{a}{b} \leq 1$ . Using the reproducing kernel property in  $\mathcal{B}_\alpha(0, 1)$  one gets

$$\int_0^\infty \frac{a}{b}\mathcal{R}_\alpha\left(\frac{a}{b}z, \frac{a}{b}t\right)\mathcal{R}_\alpha(z, t)dt = \frac{a}{b}\mathcal{R}_\alpha\left(\frac{a}{b}z, \frac{a}{b}z\right). \quad (29)$$

We just proved that, if  $a \leq b$ ,

$$\int_0^\infty ab\mathcal{R}_\alpha(ax, at)\mathcal{R}_\alpha(bx, bt)dt = a\mathcal{R}_\alpha(ax, ax) \quad (30)$$

Then, as  $b \leq b+s$ , expanding (27) and using (30) on each of the 3 terms, we get the desired equality. ■

**Lemma 3** *Let*

$$P(a, r) = \int_0^a \int_a^{a+r} \mathcal{R}_\alpha^2(t, x) dt dx.$$

and

$$Q(a, r) = \int_a^{a+r} \int_{a+r}^\infty \mathcal{R}_\alpha^2(t, x) dt dx.$$



Then, there exists constants  $K, L, K'$  and  $L'$  such that, for every  $a$  and  $r$ ,

$$P(a, r) \leq K \log r + L, \quad (31)$$

and

$$Q(a, r) \leq K' \log r + L'. \quad (32)$$

**Proof.** Consider the following integrals

$$\begin{aligned} H_1(y, u) &= \int_{y-u}^y \int_{y+u}^{\infty} \mathcal{R}_{\alpha}^2(t, x) dt dx \\ H_2(y, u) &= \int_0^{y-u} \int_y^{y+u} \mathcal{R}_{\alpha}^2(t, x) dt dx. \\ I(y, u) &= \int_{y-u}^y \int_y^{y+u} \mathcal{R}_{\alpha}^2(t, x) dt dx. \end{aligned}$$

We will prove that  $H_1(y, u)$  and  $H_2(y, u)$  are both uniformly bounded for  $y \geq u$  and that there exist constants  $K$  and  $L$  such that, for every  $y$  and  $u$  with  $y \geq u$ ,

$$I(y, u) \leq K \log u + L. \quad (33)$$

The bound on  $Q$  is then obtained by noticing that

$$Q(a, r) = H_1(a + r, r) + H_2(a + r, r).$$

To estimate  $P$  we have to separate in cases: if  $a < r$  then

$$P(a, r) \leq I(r, r) \leq K \log r + L,$$

and if  $a \geq r$  we have

$$P(a, r) = H_2(a, r) + I(a, r) \leq K \log r + L.$$

We will organize the estimates in two steps: the first one contains the estimates of  $H_1(y, u)$  and  $H_2(y, u)$  and the second one of  $I(y, u)$ .

**Step 1.** Using formulas (6) and (7) one can assure the existence of constants  $K$  and  $K'$  such that

$$\mathcal{R}_{\alpha}^2(t, x) \leq K \frac{1}{(x-t)^2} + K' \frac{1}{(x^2-t^2)^2}, \quad (34)$$

for all non-negative  $t, x$ . For  $y \leq 1$  the result easily follows from the estimate of the Trace. Let us consider  $y > 1$ . Using (34) in the definition of  $H_1(y, u)$  we obtain constants  $K$  and  $K'$  such that

$$\begin{aligned} H_1(y, u) &\leq K \int_{y-u}^y \int_{y+u}^{\infty} \left( \frac{1}{x-t} \right)^2 dx dt + K' \int_{y-u}^y \int_{y+u}^{\infty} \left( \frac{1}{x^2-t^2} \right)^2 dx dt \\ &\leq Ku \int_{y+u}^{\infty} \left( \frac{1}{x-y} \right)^2 dx + K'u \int_{y+u}^{\infty} \left( \frac{1}{x-y} \right)^2 \left( \frac{1}{x+y} \right)^2 dx \\ &= Ku \int_u^{\infty} \left( \frac{1}{\theta} \right)^2 d\theta + K'u \int_u^{\infty} \left( \frac{1}{\theta} \right)^2 \left( \frac{1}{\theta+2y} \right)^2 d\theta \\ &\leq (K + K')u \int_u^{\infty} \left( \frac{1}{\theta} \right)^2 d\theta \\ &= (K + K') \int_1^{\infty} \left( \frac{1}{\tau} \right)^2 d\tau. \end{aligned}$$

The estimate of  $H_2(y, u)$  follows the same lines.

**Step 2.** A change of variables in the double integral results in

$$I(y, u) = \int_{1-\frac{u}{y}}^1 \int_1^{1+\frac{u}{y}} y^2 \mathcal{R}_\alpha^2(yt, yx) dt dx.$$

Writing the integral explicitly and inserting asymptotic formulas (6) and (7), one sees that

$$I(y, u) = \int_{1-\frac{u}{y}}^1 \int_1^{1+\frac{u}{y}} (\mathcal{L}^y(t, x) + \mathcal{E}^y(t, x))^2 dt dx,$$

with

$$\mathcal{L}^y(t, x) = \frac{2}{\pi} \frac{\sin \eta_{yx} t \cos \eta_{yt} - \sin \eta_{yt} x \cos \eta_{yx}}{x^2 - t^2} \quad (35)$$

and  $\mathcal{E}^y(t, x) = \mathcal{O}(y^{-1})$ . As a result (and keeping in mind that  $u \leq y$ ),  $I(y, u) \leq 2\tilde{I}(y, u) + L_0$ , for some  $L_0$  independent of  $y$  and  $u$ , where

$$\tilde{I}(y, u) = \int_{1-\frac{u}{y}}^1 \int_1^{1+\frac{u}{y}} (\mathcal{L}^y(t, x))^2 dt dx.$$

Writing  $k_\alpha = -(\frac{1}{2}\alpha - \frac{1}{4})\pi$  and using (35), gives

$$\begin{aligned} \tilde{I}(y, u) &= \int_{1-\frac{u}{y}}^1 \int_1^{1+\frac{u}{y}} \left( \frac{\sin(yt + k_\alpha)x \cos(yx + k_\alpha) - \sin(yx + k_\alpha)t \cos(yt + k_\alpha)}{t^2 - x^2} \right)^2 dt dx \\ &= \int_{1-\frac{u}{y}}^1 \int_1^{1+\frac{u}{y}} \left( \frac{x \sin(y(t-x)) + (x-t) \sin(yx + k_\alpha) \cos(yt + k_\alpha)}{t^2 - x^2} \right)^2 dt dx \\ &= \int_{1-\frac{u}{y}}^1 \int_1^{1+\frac{u}{y}} \left( \frac{1}{t+x} \right)^2 \left( y \operatorname{sinc}\left(\frac{y}{\pi}(t-x)\right) - \sin(yx + k_\alpha) \cos(yt + k_\alpha) \right)^2 dt dx, \end{aligned}$$

where we are using the usual notation  $\operatorname{sinc}(x) = \sin(\pi x)/\pi x$ . From the last expression it follows that there exists positive constants  $A$  and  $B$ , independent of  $y$  and  $u$ , such that

$$\tilde{I}(y, u) \leq A \int_{1-\frac{u}{y}}^1 \int_1^{1+\frac{u}{y}} \left( y \operatorname{sinc}\left(\frac{y}{\pi}(t-x)\right) \right)^2 dt dx + B \leq A\pi^2 \mathcal{S}(y, u) + B,$$

where the second inequality is obtained doing a change of variables and writing

$$\mathcal{S}(y, u) = \int_{\frac{y-u}{\pi}}^{\frac{y}{\pi}} \int_{\frac{y}{\pi}}^{\frac{y+u}{\pi}} (\operatorname{sinc}(t-x))^2 dt dx.$$

Yet another change of variables shows that

$$\begin{aligned} \mathcal{S}(y, u) &= \mathcal{S}(u, u) \leq \int_0^{\frac{u}{\pi}} \int_{\frac{u}{\pi}}^\infty (\operatorname{sinc}(t-x))^2 dt dx + \int_0^{\frac{u}{\pi}} \int_{\mathbb{R}^-} (\operatorname{sinc}(t-x))^2 dt dx \\ &= \int_0^{\frac{u}{\pi}} \int_{\mathbb{R}} (\operatorname{sinc}(t-x))^2 dt dx - \int_0^{\frac{u}{\pi}} \int_0^{\frac{u}{\pi}} (\operatorname{sinc}(t-x))^2 dt dx \\ &= \frac{u}{\pi} - \int_0^{\frac{u}{\pi}} \int_0^{\frac{u}{\pi}} (\operatorname{sinc}(t-x))^2 dt dx, \end{aligned}$$

the last equality being true because  $\int_{\mathbb{R}} \text{sinc}(t-x)^2 dt = 1$ . Now, Landau's inequality [11, (8)] gives

$$\int_0^{\frac{u}{\pi}} \int_0^{\frac{u}{\pi}} (\text{sinc}(t-x))^2 dt dx \geq \frac{u}{\pi} - C \log\left(\frac{u}{\pi}\right) - B.$$

It follows that

$$\mathcal{S}(y, u) \leq C \log\left(\frac{u}{\pi}\right) - B.$$

Thus, for some positive constants  $K$  and  $L$  not depending on  $r$ ,

$$I(y, u) \leq K \log u + L.$$

■

### 3.2.2 Proof of Proposition 1.

**Proof.** Since the integrand is always non-negative, the double sum in (25) is bounded below by any of the single sums. Thus,

$$\text{Norm} \geq \sum_{i=1}^n \int_{b_i}^{b_i+s_i} \int_{b_i}^{b_i+s_i} [(a+r)\mathcal{R}_{\alpha}((a+r)t, (a+r)x) - a\mathcal{R}_{\alpha}(at, ax)]^2 dt dx. \quad (36)$$

Denote each of the double integrals in the sum (36) by  $N_i$ . The duality (13) between  $I$  and  $S$  gives

$$N_i = \int_a^{a+r} \int_a^{a+r} [(b_i+s_i)\mathcal{R}_{\alpha}((b_i+s_i)t, (b_i+s_i)x) - b_i\mathcal{R}_{\alpha}(b_it, b_ix)]^2 dt dx. \quad (37)$$

Set

$$\mathcal{M}_{\alpha}(x, t; b_i, s_i) = (b_i+s_i)\mathcal{R}_{\alpha}((b_i+s_i)t, (b_i+s_i)x) - b_i\mathcal{R}_{\alpha}(b_it, b_ix). \quad (38)$$

Using Lemma 2 with  $b = b_i$  and  $s = s_i$ , we get that there exists a constant  $L_0$ , not depending on  $a$  or  $r$ , such that

$$N_i \geq \frac{1}{\pi} r s_i - L_0 - \int_0^a \int_a^{a+r} \mathcal{M}_{\alpha}^2(x, t; b_i, s_i) dt dx - \int_a^{a+r} \int_{a+r}^{\infty} \mathcal{M}_{\alpha}^2(x, t; b_i, s_i). \quad (39)$$

Applying the inequality  $(a+b)^2 \leq 2(a^2 + b^2)$  to (38) gives

$$\mathcal{M}_{\alpha}^2(x, t; b_i, s_i) \leq 2(b_i+s_i)^2 \mathcal{R}_{\alpha}^2((b_i+s_i)x, (b_i+s_i)t) + 2b_i^2 \mathcal{R}_{\alpha}^2(b_ix, b_it).$$

Plugging in (39), and doing a change of variable, we get

$$\begin{aligned} N_i \geq & \frac{1}{\pi} r s_i - L_0 - 2P((b_i+s_i)a, (b_i+s_i)r) - 2P(b_ia, b_ir) \\ & - 2Q((b_i+s_i)a, (b_i+s_i)r) - 2Q(b_ia, b_ir), \end{aligned} \quad (40)$$

with,

$$P(a, r) = \int_0^a \int_a^{a+r} \mathcal{R}_{\alpha}^2(x, t) dt dx, \quad (41)$$

and

$$Q(a, r) = \int_a^{a+r} \int_{a+r}^{\infty} \mathcal{R}_\alpha^2(x, t) dt dx. \quad (42)$$

Therefore, Lemma 3, provides constants  $K_1, K_2$  and  $L_1$  not depending on  $a$  or  $r$  such that

$$N_i \geq \frac{1}{\pi} r s_i - L_0 - K_1 \log((b_i + s_i)r) - K_1 \log(b_i r) - K_2 \log((b_i + s_i)r) - K_2 \log(b_i r) - L_1. \quad (43)$$

Finally, from the inequality above we find constants  $K$  and  $L$  not depending on  $a$  or  $r$  such that

$$N_i \geq \frac{1}{\pi} r s_i - K \log(r) - L. \quad (44)$$

Plugging this on (36) we get (for new constants  $K$  and  $L$ ),

$$\text{Norm} \geq \frac{1}{\pi} r s - K \log(r) - L. \quad (45)$$

■

## 4 Sampling and interpolation and the eigenvalue problem

In this section we will prove a series of results required to connect the sampling and interpolation problem to the eigenvalue problem of the previous section. Here we will make use of the convolution structure associated to the Hankel transform.

Write  $A \lesssim B$  to signify that  $A \leq CB$  for some constant  $C > 0$ , independent of whatever arguments are involved. The next proposition is the analogue of Proposition 1 in [11].

**Proposition 2** *Let  $S$  be bounded, and let  $\{t_n\}$  be a set of interpolation for  $B_\alpha(S)$ . Then the points of  $\{t_n\}$  are separated by at least some positive distance  $d$ , and the interpolation can be performed in a stable way.*

**Proof.** From

$$f(t) = \int_S H_\alpha f(x) J_\alpha(tx) (tx)^{\frac{1}{2}} dx$$

one has

$$|f(t)|^2 \lesssim \int_0^\infty |H_\alpha f(x)|^2 dx = \int_0^\infty |f(x)|^2 dx.$$

Likewise, the identity

$$f'(t) = \int_S H_\alpha f(x) \frac{\partial J_\alpha(tx) (tx)^{\frac{1}{2}}}{\partial t} dx$$

provides a similar estimate for  $|f'(t)|$ . The rest of the proof completely follows Landau [11]. ■

Now we will prove the lemmas corresponding to Lemma 1 and Lemma 2 in [11].

**Lemma 4** *Let  $S$  be bounded and  $\{t_n\}$  a set of sampling for  $B_\alpha(S)$ , whose points are separated by at least  $2d > 0$ . Let  $I$  be any compact set,  $I^+$  be the set of points whose distance to  $I$  is less than  $d$ , and  $n(I^+)$  be the number of points of  $\{t_n\}$  contained in  $I^+$ . Then  $\lambda_{n(I^+)}(I, S) \leq \gamma < 1$ , where  $\gamma$  depends on  $S$  and  $\{t_n\}$  but not in  $I$ .*

**Proof.** To adapt the arguments of [11], we need the convolution structure associated with the Hankel transform outlined in section 2. Let  $h$  be a function with support in  $[0, d]$  such that we can find constants  $K_1, K_2$  such that its Hankel transform satisfies

$$K_1 x^{\alpha+\frac{1}{2}} \leq (H_\alpha h)(x) \leq K_2 x^{\alpha+\frac{1}{2}}, \quad (46)$$

for every  $x \in S$ . In order to construct such a function, we use the fact (see [20, pag. 482]) that, if  $A$  and  $B$  are real (not both zero) and  $\alpha > -1$ , then the function  $AJ_\alpha(z) + BzJ'_\alpha(z)$  has all its zeros on the real axis, except that it has two purely imaginary ones when  $A/B + \alpha < 0$ . Thus, if  $z_0$  is a complex number outside the imaginary and the real axis, then the function

$$x^{-\alpha-\frac{1}{2}} \mathcal{R}_\alpha(x, z_0) = z_0^{\frac{1}{2}} \frac{J_\alpha(z_0)x^{-(\alpha-1)}J'_\alpha(x) - x^{-\alpha}J_\alpha(x)z_0J'_\alpha(z_0)}{z_0^2 - x^2}$$

is bounded away from zero and infinity for every  $x \in S$ . Since

$$\mathcal{R}(x, t) = H_\alpha(\chi_{[0,1]}J_\alpha(\cdot t)(\cdot)^{1/2})(x),$$

then the function  $h$  defined via its Hankel transform as

$$(H_\alpha h)(x) = \mathcal{R}_\alpha(dx, z_0)$$

has the desired property. Now define

$$g(x) := f *_\alpha h(x) = x^{\alpha+\frac{1}{2}} \left(\frac{\pi}{2}\right)^{\frac{\alpha}{2}} \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(t) \tau_x h(t) dt.$$

Since

$$H_\alpha g(x) = H_\alpha(f *_\alpha h)(x) = x^{-\alpha-\frac{1}{2}} (2\pi)^{\frac{\alpha}{2}} H_\alpha f(x) H_\alpha h(x)$$

and  $f \in B_\alpha(S)$  then clearly also  $g \in B_\alpha(S)$ . It follows that

$$\|g\|^2 \lesssim \sum_n |g(t_n)|^2.$$

Since  $K_1 x^{\alpha+\frac{1}{2}} \leq H_\alpha h(x)$ , then

$$\|f\|^2 \lesssim \|g\|^2,$$

By formula (8),  $\text{supp } \tau_x h \subset [\max\{0, x-d\}, x+d]$  and one can write

$$\begin{aligned} |g(x)|^2 &\leq \left( x^{\alpha+\frac{1}{2}} \left(\frac{\pi}{2}\right)^{\frac{\alpha}{2}} \frac{1}{\Gamma(\alpha+1)} \int_0^\infty f(t) \tau_x h(t) dt \right)^2 \\ &\leq x^{2\alpha+1} \left( \left(\frac{\pi}{2}\right)^{\frac{\alpha}{2}} \frac{1}{\Gamma(\alpha+1)} \right)^2 \left( \int_{x-d}^{x+d} f(t) \tau_x h(t) dt \right)^2 \\ &\lesssim x^{2\alpha+1} \left( \int_{x-d}^{x+d} f(t) \tau_x h(t) dt \right)^2 \\ &\lesssim \left( \int_{x-d}^{x+d} \tau_x h(t)^2 x^{2\alpha+1} dt \right) \int_{x-d}^{x+d} |f(t)|^2 dt. \end{aligned}$$

Moreover,

$$\begin{aligned}
\int_{x-d}^{x+d} \tau_x h(t)^2 x^{2\alpha+1} dt &= x^{2\alpha+1} \|\tau_x h\|^2 \\
&= x^{2\alpha+1} \|j_\alpha(x \cdot) H_\alpha h(\cdot)\|^2 \\
&= \int_S x^{2\alpha+1} j_\alpha^2(xs) H_\alpha h(s)^2 ds \\
&= \int_S \left( (xs)^{\alpha+\frac{1}{2}} j_\alpha(xs) \right)^2 \left( \frac{H_\alpha h(s)}{s^{\alpha+\frac{1}{2}}} \right)^2 ds \\
&= C'_\alpha \int_S xs J_\alpha(xs)^2 \left( \frac{H_\alpha h(s)}{s^{\alpha+\frac{1}{2}}} \right)^2 ds \\
&\lesssim m(S) \int_S xs J_\alpha(xs)^2 ds \\
&\lesssim \int_0^{r_0} xs J_\alpha(xs)^2 ds, \text{ for some } r_0. \\
&\leq C,
\end{aligned}$$

for some constant  $C > 0$ . We have thus shown that

$$|g(x)|^2 \lesssim \int_{x-d}^{x+d} |f(t)|^2 dt. \quad (47)$$

Now we impose the  $n(I^+)$  orthogonality conditions:

$$\int_0^\infty f(t) \tau_{t_n} h(t) dt = 0$$

for every  $t_n \in I^+$ . This gives  $g(t_n) = 0$ , for every  $t_n \in I^+$ . Finally, using the separation of  $\{t_n\}$  and the definition of  $I^+$ ,

$$\begin{aligned}
\|f\|^2 &\leq \|g\|^2 \leq K \sum_{t_n \notin I^+} |g(t_n)|^2 \leq K' \sum_{t_n \notin I^+} \int_{t_n-d}^{t_n+d} |f(t)|^2 dt \leq K' \int_{\mathbb{R}^+ \setminus I} |f(t)|^2 dt \\
\frac{1}{\|f\|^2} \int_I |f(t)|^2 dt &= 1 - \frac{1}{\|f\|_2^2} \int_{\mathbb{R}^+ \setminus I} |f(t)|^2 dt \leq 1 - \frac{1}{K'} < 1.
\end{aligned}$$

Since  $K'$  is independent of  $I$ , the Lemma is proved. ■

**Lemma 5** *Let  $S$  be bounded and  $\{t_n\}$  a set of interpolation for  $B_\alpha(S)$ , whose points are separated by at least  $d > 0$ . Let  $I$  be any compact set,  $I^-$  be the set of points whose distance to the complement of  $I$  exceeds  $\frac{d}{2}$ , and  $n(I^-)$  be the number of points of  $\{t_n\}$  contained in  $I^-$ . Then  $\lambda_{n(I^-)-1}(I, S) \geq \delta > 0$ , where  $\delta$  depends on  $S$  and  $\{t_n\}$  but not in  $I$ .*

**Proof.** We have shown in Proposition 1 that the interpolation can be done in a stable way, thus

$$\|g\|^2 \leq K \sum_n |g(t_n)|^2.$$

Now, for each  $t_l$  let  $\phi_l \in B(S)$  be the interpolating function that is 1 at  $t_l$  and 0 in the rest of the  $t_n$ , all these functions are linearly independent. Let  $h$  be the same as in the proof of Lemma 4 and define  $\psi_l \in B_\alpha(S)$  by

$$(\psi_l)(x) = H_\alpha \left( \frac{(\cdot)^{\alpha+\frac{1}{2}} H_\alpha \phi_l(\cdot)}{(2\pi)^{\frac{\alpha}{2}} H_\alpha h(\cdot)} \right),$$

for every  $x \in S$  (recall that  $x^{-\alpha-\frac{1}{2}} H_\alpha h$  is bounded away from zero in  $S$ ). By Hankel transform,

$$\phi_n = \psi_n *_\alpha h.$$

Given  $f \in \text{span}\{\psi_n\}_{t_n \in I^-}$ , let  $g = f *_\alpha h$ . Then  $g$  is a linear combination of  $\phi_n$  with  $t_n \in I^-$ , thus  $g(t_n) = 0$  for  $t_n \notin I^-$ . We get, using the estimates of the proof of Lemma 4 leading to (47),

$$\|f\|^2 \leq K \|g\|^2 \leq K' \sum_{t_n \in I^-} |g(t_n)|^2 \leq K'' \sum_{t_n \in I^-} \int_{t_n-d}^{t_n+d} |f(t)|^2 dt \leq K'' \int_I |f(t)|^2 dt$$

so

$$\lambda_{k-1}(I, S) \geq \inf_{f \in \text{span}\{\psi_n\}_{t_n \in I^-}} \frac{\int_I |f(t)|^2 dt}{\int_0^\infty |f(t)|^2 dt} \geq \frac{1}{K''}.$$

Once again  $K''$  does not depend on  $I$  and we are done. ■

The information contained in the proof of the above Lemmas, namely inequality (47), allows to show that the separation implies the upper inequality in the definitions of sampling and the frame properties.

**Proposition 3** *Let  $S$  be bounded, and let  $\{t_n\}$  be a set of points separated by at least some positive distance  $d$ . Then, there exists a constant  $B$  such that, for every  $f \in B_\alpha(S)$ , we have*

$$\sum_n |g(t_n)|^2 \leq B \|g\|^2.$$

**Proof.** Let  $g \in B_\alpha(S)$  and  $h$  the function constructed in Lemma 4. Write

$$f = H_\alpha \left( \frac{(\cdot)^{\alpha+\frac{1}{2}} H_\alpha g(\cdot)}{(2\pi)^{\frac{\alpha}{2}} H_\alpha h(\cdot)} \right).$$

Clearly  $f \in B_\alpha(S)$  and by construction  $g = f *_\alpha h$ . Thus the estimate (46) gives

$$\|f\|^2 = \|H_\alpha f\|^2 \leq C \|(\cdot)^{-\alpha-\frac{1}{2}} (H_\alpha h)(\cdot) (H_\alpha f)(\cdot)\|^2 = C \|H_\alpha g\|^2 = C \|g\|^2,$$

with  $C$  independent of  $g$ . In the proof of Lemma 4 we have seen that if  $g = f *_\alpha h$  then (47) holds, allowing us to write

$$\sum_n |g(t_n)|^2 \leq C \sum_n \int_{x-d}^{x+d} |f(t)|^2 dt = C \|f\|^2 \leq C' \|g\|^2.$$

■

## 5 Proof of the main results

Let  $\{t_n\}$  be a sequence with separation constant  $d$ . We will denote by  $S^+$  the set of points whose distance to  $S$  is less than  $d$  and by  $S^-$  the set of points whose distance to the complement of  $S$  exceeds  $d$ . We will use the notation  $\lfloor x \rfloor$  to denote the largest integer smaller or equal than  $x$ .

Using the identities (13) and (14) one can see that  $\lambda_{k-1}(r, S)$  is also the  $k^{\text{th}}$  eigenvalue of the problem of concentrating on the set  $rS$  the functions whose Hankel transform is supported in  $[0, 1]$ . The sampling theorem associated with the Hankel transform [9] (together with the [8, (2.3)]) states that  $\{j_{\alpha, n}\}$  is a sequence of both sampling and interpolation, and is known to be a perturbation of the set  $\{\frac{n}{\pi}\}$ . Then, there exists  $\Upsilon = \mathcal{O}(1)$  such that, when  $S$  is the union of  $N$  intervals, the number of these points contained in  $S^+$  is at most  $\lfloor \frac{1}{\pi} r m(S) \rfloor + \Upsilon N$  and their number in  $S^-$  is at least  $\lfloor \frac{1}{\pi} r m(S) \rfloor - \Upsilon N$ . Now, from Lemma 5 and Lemma 6 of Section 5, there are  $\gamma_0, \delta_0$  such that

$$\lambda_{\lfloor \frac{1}{\pi} r m(S) \rfloor + \Upsilon N}(r, S) \leq \gamma_0 < 1 \quad (48)$$

$$\lambda_{\lfloor \frac{1}{\pi} r m(S) \rfloor - \Upsilon N - 1}(r, S) \geq \delta_0 > 0. \quad (49)$$

**Theorem 3 (Sampling)** *Let  $S$  be a finite union of intervals. If  $\{t_n\}$  is a set of sampling for  $B_\alpha(S)$ , then  $[0, r]$  must contain at least  $(\frac{1}{\pi} r m(S) - A \log r - B)$  points of  $\{t_n\}$ , with  $A$  and  $B$  constants not depending on  $r$ .*

**Proof.** Let  $\{t_n\}$  is a set of sampling for  $B_\alpha(S)$ . By Lemma 4 there exists  $\gamma$  independent of  $r$  such that

$$\lambda_{n(I)+2} \leq \lambda_{n(I^+)} \leq \gamma < 1.$$

From (49),

$$\lambda_{\lfloor \frac{1}{\pi} r m(S) \rfloor - \Upsilon N - 1}(r, S) \geq \delta_0 > 0.$$

As the number of eigenvalues between  $\delta_0$  and  $\gamma$  increase at most logarithmically with  $r$ , we have

$$\lfloor \frac{1}{\pi} r m(S) \rfloor - \Upsilon N - 1 - n(I) + 2 \leq A' \log r + B'.$$

Thus,

$$n(I) \geq \frac{1}{\pi} r m(S) - A \log r - B,$$

for some  $A$  and  $B$  not depending on  $r$ . □ ■

**Theorem 4 (Interpolation)** *Let  $S$  be a finite union of intervals. If  $\{t_n\}$  is a set of interpolation for  $B_\alpha(S)$ , then  $[0, r]$  must not contain more than  $(\frac{1}{\pi} r m(S) - C \log r - D)$  points of  $\{t_n\}$ , with  $C$  and  $D$  constants not depending on  $r$ .*

**Proof.** Let  $\{t_n\}$  be a set of interpolation for  $B_\alpha(S)$ . By Lemma 5 there exists  $\delta$  independent of  $r$  such that

$$\lambda_{n(I)-3} \geq \lambda_{n(I^-)} \geq \delta > 0.$$

From (48),

$$\lambda_{\lfloor \frac{1}{\pi} r m(S) \rfloor + \Upsilon N}(r, S) \leq \gamma_0 < 1.$$



As the number of eigenvalues between  $\delta$  and  $\gamma_0$  increase at most logarithmically with  $r$  we have

$$(n(I) - 3) - \left( \left\lfloor \frac{1}{\pi} r m(S) \right\rfloor + \Upsilon N \right) \leq C' \log r + D'$$

Thus,

$$n(I) \leq \frac{1}{\pi} r m(S) + C \log r + D,$$

for constants  $C, D$  not depending on  $r$ . □ ■

To extend the result to more general sets, we can proceed as in Landau [10, pag. 49-50]. In the sampling case, the result can be extended to a general measurable set  $S$  by observing that it suffices to prove the result for compact sets and then cover a compact  $S$  set by a finite collection of intervals with disjoint interiors and measure arbitrary close to the measure of  $S$ . In the interpolation case, the result is extended to bounded measurable sets by approximating in measure from the outside by bounded open sets.

Finally we remark that the case of functions whose Hankel transform is supported on  $[a, a + r]$  cannot be reduced to a case where a sequence of both sampling and interpolation is known to exist (since, unlike the Fourier case, our eigenvalue problem is not translation invariant). Nevertheless, we can still obtain asymptotic versions of the inequalities which can be used to prove Theorem 1 and Theorem 2. This problem is also present in [12], who offers a solution which can be adapted to our setting. From Lemma 4 we know that

$$\#\{\lambda_j(I, S) > \gamma\} \leq n(I^+) \leq n(I) + o(r), \quad r \rightarrow \infty.$$

and using exactly the same argument of [12, page 582], we can obtain the lower estimate

$$\#\{\lambda_j(I, S) > \gamma\} \geq \text{Trace} - \frac{1}{1 - \gamma} (\text{Trace} - \text{Norm}).$$

Thus,

$$n(I) \geq \frac{1}{\pi} r m(S) - \frac{1}{1 - \gamma} A \log r - B - o(r) \quad r \rightarrow \infty.$$

The estimate required for interpolation can be obtained in a similar way. Now, Theorem 1 and Theorem 2 are straightforward consequences of the definitions of lower and upper density.

## Acknowledgments

We would like to thank Richard Laugesen, Óscar Ciaurri, Juan Luis Varona for discussions related to the problem treated in this work and to thank Kristian Seip for pointing us reference [12]. We would also like to thank the referee for suggesting several improvements and to the referees of an earlier version of the paper, who hold the paper to a higher standart by challenging us to prove the stronger “translation-invariant” results contained in this version.

## References

- [1] S. Bergman. *The Kernel Function and Conformal Mapping*. Mathematical Surveys and Monographs. American Mathematical Society, 1970.

- [2] R. P. Boas and H. Pollard. Complete sets of Bessel and Legendre functions. *Ann. of Math.*, 48:366–384, 1947.
- [3] Ó. Ciaurri and L. Roncal. Littlewood-Paley-Stein  $g_k$ -functions for Fourier-Bessel expansions. *J. Funct. Anal.*, 258:2173–2204, 2010.
- [4] Ó. Ciaurri and K. Stempak. Transplantation and multiplier theorems for Fourier-Bessel expansions. *Trans. Amer. Math. Soc.*, 358:4441–4465, 2006.
- [5] I. Daubechies. *Ten Lectures on Wavelets*. CBMS-NSF Regional Conference Series in Applied Mathematics. SIAM, 1992.
- [6] K. Gröchenig, G. Kutyniok, and K. Seip. Landau’s necessary density conditions for LCA groups. *J. Funct. Anal.*, 255:1831–1850, 2008.
- [7] K. Gröchenig and H. Razafinjatovo. On Landau’s necessary density conditions for sampling and interpolation of band-limited functions. *J. London Math. Soc.*, 54:557–565, 1996.
- [8] J. J. Guadalupe, M. Perez, F. J. Ruiz, and J. L. Varona. Mean and weak convergence of Fourier-Bessel series. *J. Math. Anal. Appl.*, 173:370–389, 1993.
- [9] J. R. Higgins. An interpolation series associated with the Bessel-Hankel transform. *J. London Math. Soc.*, 5:707–714, 1972.
- [10] H. J. Landau. Necessary density conditions for sampling and interpolation of certain entire functions. *Acta Math.*, 117:37–52, 1967.
- [11] H. J. Landau. Sampling, data transmission, and the Nyquist rate. *Proceedings of the IEEE*, 55:1701–1706, 1967.
- [12] J. Marzo. Marcinkiewicz-Zygmund inequalities and interpolation by spherical harmonics. *J. Funct. Anal.*, 250:559–587, 2007.
- [13] J. Ortega-Cerdá and K. Seip. Fourier frames. *Ann. of Math.*, 155:789–806, 2002.
- [14] J. Ramanathan and T. Steger. Incompleteness of sparse coherent states. *Appl. Comput. Harmon. Anal.*, 2:148–153, 1995.
- [15] H. Rauhut. *Time-frequency and wavelet analysis of functions with symmetry properties*. PhD thesis, Technical University of Munich, 2004.
- [16] H. Rauhut and M. Rösler. Time-frequency and wavelet analysis of functions with symmetry properties. *Constr. Approx.*, 22:193–218, 2005.
- [17] M. D. Rawn. On nonuniform sampling expansions using entire interpolating functions, and on the stability of bessel-type sampling expansions. *IEEE Trans. Inform. Theory*, 35:549–557, 1989.
- [18] K. Seip. Reproducing formulas and double orthogonality in Bargmann and Bergman spaces. *SIAM J. Math. Anal.*, 22:856–876, 1991.

- [19] C. A. Tracy and H. Widom. Level spacing distributions and the Bessel kernel. *Comm. Math. Phys.*, 161:289–309, 1994.
- [20] G. N. Watson. *A Treatise on the Theory of Bessel Functions*. Cambridge University Press, Cambridge, second edition, 1944.
- [21] R. M. Young. *An Introduction to Nonharmonic Fourier Series*. Revised first edition. Academic Press, San Diego, 2001.